THE MAXIMUM LIKELIHOOD ESTIMATION OF THE PARAMETER OF THE TRUNCATED CENSORED POISSON DISTRIBUTION

By S. A. Patil

University of Poona

1. Introduction

The truncated and Censored Poisson distributions have been thoroughly studied by Cohen (1954), Moore (1952), Rider (1953) and many others. The censored truncated *chi*-square distribution has been recently studied by Perayya Sastry (1960). In this paper the truncated censored ($T_A C_B$ truncated at A and censored at B) Poisson distribution is considered. In the truncated Poisson distribution many times it is impossible to take individual observations in one or more classes. Only the number of observations lying in these classes is separately or totally known in the untruncated region. This distribution is Truncated Censored distribution. In the study of bacteria if 'λ' denotes the total number of individuals in the whole solution divided by the total volume occupied, then 'λ' gives mean density. The probability that in a sample of one unit volume there will be x individuals is given by Poisson distribution. Here zero class is completely missing. So the distribution is truncated below. The observations above some fixed number are not measured; they are pooled so the total number in above classes is known and the distribution is censored above. In a recent paper by Cohen (1960) a remark was made that in Poisson distribution of number defected per unit, zero class in absent and observations in class one are erratic and therefore he modified Poisson distribution. In this example if we can know the number of observations falling in class one then this can be treated as truncated censored below and can be studied as given below. The number of deaths from the kick of a horse has a Poisson distribution. In this problem zero class of the variable is missing. At the lower classes 1 or 2 of the variable, some of the observations terminate before observations begin. In this case resulting sample has truncated censored distribution. In the case of distribution of radioactive element, zero class is missing and the observations in above cases are not taken or they are pooled, as time required is very large. Here the resulting distribution of the sample is truncated censored Poisson distribution.

Of the truncated censored distribution the following cases are discussed : (I) Truncated censored below and truncated censored above when the number in each censored region is separately known. Its four particular cases are (1) Truncated censored below, (2) Truncated below and censored above, (3) Truncated censored above, (4) Truncated above and censored below. (II) Truncated censored above and below when total number in two censored regions is known. In each case the maximum likelihood estimating equation is obtained, which can be solved by linear interpolation. Also asymptotic variances of the estimates have been obtained. Numerical examples of these have been given and variances have been calculated.

(I) *Truncated Censored Below and Truncated Censored Above when the Number in Each Censored Region is Separately Known*

The distribution is truncated below at x_0 and above at x_1 in the untruncated part it is censored below at y_0 and above at y_1. The density function in the closed range (y_0, y_1)* is defined as

$$f_t(x) = \frac{e^{-m} m^x}{x!\,[p(x_0) - p(x_1 + 1)]} \qquad y_0 \leqslant x \leqslant y_1 \qquad (1)$$

where

$$p(x_0) = \sum_{x=x_0}^{\infty} \frac{e^{-m} m^x}{x!}$$

with similar meaning to $p(y_0), p(y_1), p(y_1 + 1)$, etc. The probability in the interval (x_0, y_0) is defined as

$$Pr(x_0 \leqslant x < y_0) = \frac{p(x_0) - p(y_0)}{p(x_0) - p(x_1 + 1)} \qquad (2)$$

and that in the interval $[y_1, x_1)$ is given by

$$Pr(y_1 < x \leqslant x_1) = \frac{p(y_1 + 1) - p(x_1 + 1)}{p(x_0) - p(x_1 + 1)}. \qquad (3)$$

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* A closed interval is denoted by (a, b) semi-open interval by $[a, b)$ or $(a, b]$ and open interval by $[a, b)$.

For the estimation of the parameter 'm' let $x_1, x_2, \cdots, x_n$ be measured sample from (y_0, y_1) and n_0 observations lie in $(x_0, y_0]$ and n_1 observations lie in $[y_1, x_1)$. The likelihood L is

$$L = \frac{e^{-nm} \sum_{i=1}^{n} x_i \, [p(y_1+1) - p(x_1+1)]^{n_1} \, [p(x_0) - p(y_0)]^{n_0}}{x_1! \, x_2! \ldots x_n! \, [p(x_0) - p(x_1+1)]^{n+n_0+n_1}} . \tag{4}$$

Taking logarithm of (4) and differentiating w.r.t. 'm' the estimating equation is obtained as

$$\frac{1}{n}\frac{d \log L}{dm} = -1 + \frac{\bar{x}}{m} + \frac{n_1}{n}\left[\frac{f(y_1) - f(x_1)}{p(y_1+1) - p(x_1+1)}\right] + \frac{n_0}{n}\,\frac{f(x_0-1) - f(y_0-1)}{p(x_0) - p(y_0)} - \left(\frac{n+n_0+n_1}{n}\right) \times \frac{f(x_0-1) - f(x_1)}{p(x_0) - p(x_1+1)} = 0 \tag{5}$$

where

$$\bar{x} = \sum_{i=1}^{n} x_i$$

and

$$\frac{d}{dm} p(x_0) = f(x_0 - 1)$$

where

$$f(x_0) = \frac{e^{-m} m^{x_0}}{x_0!} .$$

For variance the second derivative of (4) is

$$\frac{1}{n}\frac{d^2 \log L}{dm^2} = -\frac{\bar{x}}{m^2} + \frac{n_1}{n}\left\{-\left[\frac{f(y_1) - f(x_1)}{p(y_1+1) - p(x_1+1)}\right]^2 + \frac{f(y_1-1) - f(y_1) - f(x_1-1) + f(x_1)}{p(y_1+1) - p(x_1+1)}\right\} + \frac{n_0}{n}\left\{-\left[\frac{f(x_0-1) - f(y_0-1)}{p(x_0) - p(y_0)}\right]^2 + \frac{f(x_0-2) - f(x_0-1) - f(y_0-2) + f(y_0-1)}{p(x_0) - p(y_0)}\right\} - \left(\frac{n+n_0+n_1}{n}\right)$$

$$\times \left\{ - \left[\frac{f(x_0 - 1) - f(x_1)}{p(x_0) - p(x_1 + 1)} \right]^2 + \frac{f(x_0 - 2) - f(x_0 - 1) - f(x_1 - 1) + f(x_1)}{p(x_0) - p(x_1 + 1)} \right\} \tag{6}$$

and taking expection of

$$\frac{1}{n} \frac{d^2 \log L}{dm^2}.$$

$$\frac{1}{n} \mathscr{E} \left(\frac{d^2 \log L}{dm^2} \right)$$

$$= -\frac{1}{m} \frac{p(y_0 - 1) - p(y_1 - 1)}{p(x_0) - p(x_1 + 1)} + \frac{n_1}{n} \left\{ - \left[\frac{f(y_1) - f(x_1)}{p(y_1 + 1) - p(x_1 + 1)} \right]^2 \right.$$

$$\left. + \frac{f(y_1 - 1) - f(y_1) + f(x_1) - f(x_1 - 1)}{p(y_1 + 1) - p(x_1 + 1)} \right\} + \frac{n_0}{n} \left\{ - \left[\frac{f(x_0 - 1) - f(y_0 - 1)}{p(x_0) - p(y_0)} \right]^2 \right.$$

$$\left. + \frac{f(x_0 - 2) - f(x_0 - 1) + f(y_0 - 2) - f(y_0 - 1)}{p(x_0) - p(y_0)} \right\} - \left(\frac{n + n_0 + n_1}{n} \right)$$

$$\times \left\{ - \left[\frac{f(x_0 - 1) - f(x_1)}{p(x_0) - p(x_1 + 1)} \right]^2 + \frac{f(x_0 - 2) - f(x_0 - 1) - f(x_1 - 1) + f(x_1)}{p(x_0) - p(x_1 + 1)} \right\} \tag{7}$$

where $\mathscr{E}(\ )$ denotes expected value of bracketed quantity.

Particular cases of this distribution are:

Case 1. *Truncated Censored Below*

The Poisson distribution is truncated at the known point x_0. In the untruncated region it is censored below at the known point y_0. The total number of censored observations n_0 lies in $(x_0, y_0]$. Taking $n_1 = 0$ and $x_1 \to \infty$ also $y_1 \to \infty$ in [I] above we obtain the estimating equation in this case as

$$-1 + \frac{\bar{x}}{m} + \frac{n_0}{n} \left[\frac{f(x_0 - 1) - f(y_0 - 1)}{p(x_0) - p(y_0)} \right] - \left(\frac{n + n_0}{n} \right)$$

$$\times \frac{f(x_0 - 1)}{p(x_0)} = 0. \tag{8}$$

For variance the second derivative of the likelihood function reduces to

$$\frac{1}{n} \frac{d^2 \log L}{dm^2}$$

$$= -\frac{\bar{x}}{m^2} + \frac{n_0}{n} \left\{ - \left[\frac{f(x_0 - 1) - f(y_0 - 1)}{p(x_0) - p(y_0)} \right]^2 \right.$$

$$+ \frac{f(x_0-2)-f(x_0-1)-f(y_0-2)+f(y_0-1)}{p(x_0)-p(y_0)}\Big\} - \left(\frac{n+n_0}{n}\right)$$
$$\times \left\{-\left[\frac{f(x_0-1)}{p(x_0)}\right]^2 + \frac{f(x_0-2)-f(x_0-1)}{p(x_0)}\right\} \quad (9)$$

and

$$\frac{1}{n}\mathscr{E}\left(\frac{d^2 \operatorname{Log} L}{dm^2}\right)$$
$$= -\frac{1}{m}\frac{p(y_0-1)}{p(x_0)} + \frac{n_0}{n}\Big\{-\left[\frac{f(x_0-1)-f(y_0-1)}{p(x_0)-p(y_0)}\right]^2$$
$$+ \frac{f(x_0-2)-f(x_0-1)-f(y_0-2)+f(y_0-1)}{p(x_0)-p(y_0)}\Big\} - \left(\frac{n+n_0}{n}\right)$$
$$\times \left\{-\left[\frac{f(x_0-1)}{p(x_0)}\right]^2 + \frac{f(x_0-2)-f(x_0-1)}{p(x_0)}\right\}. \quad (10)$$

Case 2. *Truncated Below and Censored Above*

Here distribution is truncated below at x_0 and in the untruncated region it is censored above at y_1. The measured observations lie in (x_0, y_0).

Here taking $y_0 = x_0$ and $x_1 \to \infty$ and $n_0 = 0$ in [I] above the estimating equation reduces to

$$-1 + \frac{\bar{x}}{m} + \frac{n_1}{n}\frac{f(y_1)}{p(y_1+1)} - \left(\frac{n+n_1}{n}\right)\frac{f(x_0-1)}{p(x_0)} = 0 \quad (11)$$

and second derivative w.r.t. 'm' of the likelihood function becomes

$$\frac{1}{n}\left(\frac{d^2 \log L}{dm^2}\right)$$
$$= -\frac{\bar{x}}{m^2} + \frac{n_1}{n}\left\{-\left[\frac{f(y_1)}{p(y_1+1)}\right]^2 + \frac{f(y_1-1)-f(y_1)}{p(y_1+1)}\right\} - \left(\frac{n+n_1}{n}\right)$$
$$\times \left\{-\left[\frac{f(x_0-1)}{p(x_0)}\right]^2 + \frac{f(x_0-2)-f(x_0-1)}{p(x_0)}\right\} \quad (12)$$

and

$$\frac{1}{n}\mathscr{E}\left(\frac{d^2 \log L}{dm^2}\right)$$
$$= -\frac{1}{m}\left[\frac{p(x_0-1)-p(y_1-1)}{p(x_0)}\right] + \frac{n_1}{n}\left\{-\left[\frac{f(y_1)}{p(y_1+1)}\right]^2 + \frac{f(y_1-1)-f(y_1)}{p(y_1+1)}\right\}$$
$$-\left(\frac{n+n_1}{n}\right)\left\{-\left[\frac{f(x_0-1)}{p(x_0)}\right]^2 + \frac{f(x_0-2)-f(x_0-1)}{p(x_0)}\right\}. \quad (13)$$

Case 3. *Truncated Censored Above*

Here distribution is truncated above at x_1 and in the untruncated region it is censored at y_1, measured n observations lie in $(0, y_1)$.

Taking $x_0 = y_0 = 0$ and $n_0 = 0$ in [I] above. The estimating equation is reduced to the form

$$-1 + \frac{\bar{x}}{m} + \left(\frac{n + n_1}{n}\right) \frac{f(x_1)}{1 - p(x_1 + 1)} + \frac{n_1}{n} \frac{f(y_1) - f(x_1)}{p(y_1 + 1) - p(x_1 + 1)} = 0. \quad (14)$$

The second derivative w.r.t. 'm' of likelihood function takes the form

$$\frac{1}{n} \frac{d^2 \log L}{dm^2}$$

$$= -\frac{\bar{x}}{m^2} + \frac{n_1}{n} \left\{ + \left[\frac{f(y_1) - f(x_1)}{p(y_1 + 1) - p(x_1 + 1)} \right]^2 \right.$$

$$\left. + \frac{f(y_1 - 1) - f(y_1) - f(x_1 - 1) + f(x_1)}{p(y_1 + 1) - p(x_1 + 1)} \right\} + \left(\frac{n + n_1}{n}\right)$$

$$\times \left\{ \left[\frac{f(x_1)}{1 - p(x_1 + 1)} \right]^2 + \frac{f(x_1 - 1) - f(x_1)}{1 - p(x_1 + 1)} \right\} \quad (15)$$

and

$$\frac{1}{n} \mathscr{E} \left(\frac{d^2 \log L}{dm^2} \right)$$

$$= -\frac{1}{m} \left[\frac{1 - p(y_1 - 1)}{1 - p(x_1 - 1)} \right] + \frac{n_1}{n} \left\{ - \left[\frac{f(y_1) - f(x_1)}{p(y_1 + 1) - p(x_1 + 1)} \right]^2 \right.$$

$$\left. + \frac{f(y_1 - 1) - f(y_1) + f(x_1) - f(x_1 - 1)}{p(y_1 + 1) - p(x_1 + 1)} \right\} + \left(\frac{n + n_1}{n}\right)$$

$$\times \left\{ \left[\frac{f(x_1)}{1 - p(x_1 + 1)} \right]^2 + \frac{f(x_1 - 1) - f(x_1)}{1 - p(x_1 + 1)} \right\}. \quad (16)$$

Case 4. *Truncated Above and Censored Below*

Here distribution is truncated above at x_1 and censored below at y_0. The known observations lie in (y_0, x_1). Taking $x_0 = 0$, $y_1 = x_1$ and $n_1 = 0$ in [I] we obtain the estimating equation as

$$-1 + \frac{\bar{x}}{m} + \frac{n_0}{n} \left[\frac{-f(y_0 - 1)}{1 - p(y_0)} \right] + \left(\frac{n + n_0}{n}\right) \frac{f(x_1)}{1 - p(x_1 + 1)} = 0 \quad (17)$$

and second derivative w.r.t. 'm' of the likelihood function reduces to

$$\frac{1}{n}\frac{d^2\log L}{dm^2}$$

$$= -\frac{\bar{x}}{m^2} - \frac{n_0}{n}\left\{\left[\frac{f(y_0-1)}{1-p(y_0)}\right]^2 + \frac{f(y_0-2)-f(y_0-1)}{1-p(y_0)}\right\} + \left(\frac{n+n_0}{n}\right)$$

$$\times\left\{\left[\frac{f(x_0)}{1-p(x_0-1)}\right]^2 + \frac{f(x_1-1)-f(x_1)}{1-p(x_1+1)}\right\} \tag{18}$$

and

$$\frac{1}{n}\mathscr{E}\left(\frac{d^2\log L}{dm^2}\right)$$

$$= -\frac{1}{m}\left[\frac{p(y_0-1)-p(x_1-1)}{1-p(x_1+1)}\right] - \frac{n_0}{n}\left\{\left[\frac{f(y_0-1)}{1-p(y_0)}\right]^2 + \frac{f(y_0-2)-f(y_0-1)}{1-p(y_0)}\right\}$$

$$+\left(\frac{n+n_0}{n}\right)\left\{\left[\frac{f(x_0)}{1-p(x_0+1)}\right]^2 + \frac{f(x_1-1)-f(x_1)}{1-p(x_1+1)}\right\}. \tag{19}$$

(II) *Truncated Censored Above and Below when Total Number in Two Censored Regions is Known*

The distribution is truncated below at x_0, and above at x_1. In the untruncated region it is censored below at y_0 and above at y_1. The unmeasured observations in the two censored regions together is n'.

The density function in (y_0, y_1) is given by

$$f_t(x) = \frac{e^{-m}m^x}{x!\,[p(x_0)-p(x_1+1)]} \qquad y_0 \leqslant x \leqslant y_1. \tag{20}$$

The probability in the censored region is

$$Pr(x_0 \leqslant x < y_0,\ y_1 < x \leqslant x_1) = \frac{p(y_1+1)-p(x_1+1)+p(x_0)-p(y_0)}{p(x_0)-p(x_1+1)}. \tag{21}$$

For estimation, let $x_1, x_2, \cdots, x_n$ be the sample from (y_0, y_1) and n' observations in censored regions. The likelihood is given by

$$L = \frac{e^{-nm}\, m^{\sum_{i=1}^{n} x_i}\,[p(y_1+1)-p(x_1+1)-p(x_0)-p(y_0)]^{n'}}{x_1!\,x_2!\cdots,x_n!\ \ [p(x_0)-p(x_1+1)]^{n+n'}}, \tag{22}$$

The estimating equations are

$$-1+\frac{\bar{x}}{m}+\frac{n'}{n}\left\{\frac{f(y_1)-f(x_1)+f(x_0-1)-f(y_0-1)}{p(y_1+1)-p(x_1+1)+p(x_0)-p(y_0)}\right\}-\left(\frac{n+n'}{n}\right)$$

$$\times\left[\frac{f(x_0-1)-f(x_1)}{p(x_0)-p(x_1+1)}\right]=0. \tag{23}$$

For variance the second derivative of (22) is

$$\frac{1}{n}\frac{d^2\log L}{dm^2}$$

$$=-\frac{\bar{x}}{m^2}+\frac{n'}{n}\left\{-\left[\frac{f(y_1)-f(x_1)+f(x_0-1)-f(y_0-1)}{p(y_1+1)-p(x_1+1)+p(x_0)-p(y_0)}\right]^2\right.$$

$$\left.+\left[\frac{f(y_1-1)-f(y_1)-f(x_1-1)+f(x_1)+f(x_0-2)-f(x_0-1)-f(y_0-2)+f(y_0-1)}{p(y_1+1)-p(x_1+1)+p(x_0)-p(y_0)}\right]\right\}$$

$$-\left(\frac{n+n'}{n}\right)\left\{-\left[\frac{f(x_0-1)f(x_1)}{p(x_0)-p(x_1+1)}\right]^2+\frac{f(x_0-2)-f(x_0-1)-f(x_1-1)+f(x_1)}{p(x_0)-p(x_1+1)}\right\} \tag{24}$$

and taking expection of $(d^2\log L/dm^2)$

$$\frac{1}{n}\mathscr{E}\left(\frac{d^2\log L}{dm^2}\right)$$

$$=-\frac{1}{m}\left[\frac{p(y_0-1)-p(y_1+1)}{p(x_0)-p(x_1+1)}\right]+\frac{n'}{n}\left\{-\left[\frac{f(y_1)-f(x_0-1)+f(x_1)-f(y_0-1)}{p(y_1+1)-p(x_1+1)+p(x_0)-p(y_0)}\right]^2\right.$$

$$\left.+\left[\frac{f(y_1-1)-f(y_1)-f(x_1-1)+f(x_1)+f(x_0-2)-f(x_0-1)-f(y_0-2)+f(y_0-1)}{p(y_1+1)-p(x_1+1)+p(x_0)-p(y_0)}\right]\right\}$$

$$-\left(\frac{n+n'}{n}\right)\left\{-\left[\frac{f(x_0-1)-f(x_1)}{p(x_0)-p(x_1+1)}\right]^2+\frac{f(x_0-2)-f(x_0-1)-f(x_1-1)+f(x_1)}{p(x_0)-p(x_1+1)}\right\}. \tag{25}$$

For finding the solution of the estimating equations we use the method of inverse linear interpolation taking mean as an approximate solution of (5), (8), (11), (14), (17) and (23). The variance of the estimator $\hat{m}$ is obtained by the relation, as done by Cohen (1954),

$$\text{Var}(\hat{m})=\left[-\frac{d^2\log L}{dm^2}\right]^{-1}_{m=\hat{m}} \tag{26}$$

or by the relation, as done by Rider (1953),

$$\text{Var}(\hat{m})=\left[-\mathscr{E}\left(\frac{d^2\log L}{dm^2}\right)\right]^{-1}_{m=\hat{m}}. \tag{27}$$

2. Numerical Illustration

We shall take the data used by Cohen (1954) and modifying so as to suit truncation.

(I) *Truncated Censored Below and Truncated Censored Above*
Table of Observations

x per Interval	Unmeasured Number of observations in 1 Class	2	3	4	5	6	7	Observations in Class 8 unmeasured
Number of Intervals ..	203	383	525	532	408	273	139	45

Here $x_0 = 1$, $y_0 = 2$, $x_1 = 8$, and $y_1 = 7$
$n_0 = 203$, $n_1 = 45$, $n = 2260$.
We find the mean $\bar{x} = 4 \cdot 035398$.
The m.l.e. $\hat{m} = 3 \cdot 885005$.
Var from (26) Var $(\hat{m}) = 0 \cdot 00194283$ and
Var from (27) Var $(\hat{m}) = 0 \cdot 002605200$.

Case 1. *Truncated Censored Below*

Per Interval	Unmeasured Number of observations in 1 Class	2	3	4	5	6	7	8	9 and above
	203	383	525	532	408	273	139	45	43

Here $x_0 = 1$, $y_0 = 2$, $n_0 = 203$, $n = 2348$.
We find approximate mean $\bar{x} = 4 \cdot 2022998$.
Using this we get m.l.e. $\hat{m} = 3 \cdot 865233$.
Var of $(\hat{m})$ from (26) Var $(\hat{m}) = 0 \cdot 00161769$.
Var of $(\hat{m})$ from (27) Var $(\hat{m}) = 0 \cdot 001769200$.

Case 3. *Truncated Censored Above*

Per Interval	0	1	2	3	4	5	6	7	Unmeasured Number in Class 8
Number of Intervals ..	57	203	383	525	532	408	273	139	45

Taking $x_1 = 8$, $y_1 = 7$, $n_1 = 45$ and $n = 2520$.

The mean $\bar{x} = 3\cdot6996031$ and m.l.e. $\hat{m} = 3\cdot879377$.

The variance from (26) is Var $(\hat{m}) = 0\cdot00175871$.

The variance from (27) is Var $(\hat{m}) = 0\cdot00208017$.

(II) *Truncated Censored Above and Below when Total Number in Two Censored Regions is Known*

Per Interval	2	3	4	5	6	7	Number of unmeasured observations in 1 and 8 classes
Number of Intervals	385	525	532	408	273	139	248

Here $x_0 = 1$, $y_0 = 2$, $y_1 = 7$, $x_1 = 8$, $n' = 248$ and $n = 2260$.

We find mean $\bar{x} = 4\cdot035398$ and m.l.e. $\hat{m} = 3\cdot969516$ and

Variance from (26) Var $(\hat{m}) = 0\cdot00289973$ and

Variance from (27) Var $(\hat{m}) = 0\cdot00314106$.

We can compare the variance from this table. The variance in row IV is more than that in row III as separate number of observations is known in Case III but in Case IV total number of observation in two regions is known. Also the variances in column (6) are greater than those in (5) so the variance obtained by taking expections are not smaller than those obtained by replacing 'm' by its m.l.e. value $\hat{m}$ in the second order derivatives.

Table showing m.l.e.'s and its variances

Name of Distribution	No. of obs. used	Mean $\bar{x}$	m.l.e. $\hat{m}$	Var. from (26) Var. ($\hat{m}$)	Var. from (27) Var. ($\hat{m}$)
(1) Truncated censored below	2348	4·2022998	3·865233	0·00161769	0·00176920
(2) Truncated censored above	2520	3·6996031	3·879377	0·00175871	0·00208017
(3) Truncated censored below and above	2260	4·035398	3·885005	0·00194283	0·00260520
(4) Truncated censored below and above when total number in two censored regions	2260	4·035398	3·969516	0·00290423	0·00449515

3. SUMMARY

Maximum likelihood estimators are obtained for the parameter of the Poisson Distribution when it is truncated censored at known points. Six cases are considered and in each case the expression for Asymptotic variances of the estimator is obtained. A numerical example is cited to illustrate the method.

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REFERENCES

1. Cohen, A. C., Jr. .. *Jour. Amer. Stat. Ass.*, 1954, **49**, 158–68.
2. ———— .. *Biometrika*, 1955, **42**, 516–19.
3. ———— .. *Jour. Amer. Stat. Ass.*, 1960, **55**, 342–48.
4. Moore, P. G. .. *Biometrika*, 1952, **39**, 247–51.
5. Rider, Paul R. .. *Jour. Amer. Stat. Ass.*, 1953, **48**, 826–30.
6. Perayya Sastry, M. .. *Cal. Stat. Ass. Bull.*, 1960, **10**, 41–47.